

α -Cut PDE-Constrained Optimization for Patient-Specific Pulse-Wave Propagation in Elastic Arteries

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ABSTRACT

Pulse-wave analysis underpins vascular diagnosis and endovascular planning, yet patient-specific calibration is hampered by epistemic uncertainty in vessel properties, boundary data, and afterload. We present an α -cut PDE-constrained optimization framework that fits a 1D elastic-artery hemodynamic model to clinical waveforms while propagating uncertainty from fuzzy priors on wall stiffness, geometry, viscosity, and Windkessel elements. At each α -level, the feasible set U_α defines nested parameter boxes; a simultaneous multi-scenario program is solved with adjoint-based gradients and projected quasi-Newton/SQP, yielding nominal estimates and α -indexed prediction bands for pressure/flow, wall-shear surrogates, and pulse-wave velocity (PWV). A worked, physiologic example demonstrates the pipeline: automated diastolic-tail fitting recovers afterload time constant τ and R_1 - C - R_2 ; geometry-derived Moens-Korteweg PWV falls within 5.37–5.94 m/s along a 30 cm segment; inlet pressure is reproduced with RMSE = 3.61 mmHg and peak-timing error = 9 ms. α -robust calibration tightens uncertainty envelopes as α increases (e.g., $\pm 10\% \rightarrow \pm 3\%$ amplitude bands from $\alpha = 0.2$ to 0.8) and stabilizes parameters that are otherwise weakly identifiable. The method is computationally tractable (forward + adjoint per scenario), seamlessly integrates with standard vascular modeling tools, and produces clinician-interpretable bands that support threshold-based decisions (e.g., PWV cut-offs, peri-procedural pressure limits). Extensions to viscoelastic walls, type-2 fuzzy sets, and 3D–1D coupling are straightforward within the same α -cut/adjoint structure.

KEYWORDS: fuzzy uncertainty; α -cuts; PDE-constrained calibration; hemodynamic modeling; pulse-wave velocity; Windkessel identification; adjoint optimization; uncertainty quantification.

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INTRODUCTION

Pulse-wave analysis in elastic arteries underpins vascular diagnosis and therapy planning, yet patient-specific modeling is challenged by uncertain vessel and blood properties (e.g., elastic modulus E , wall thickness h , reference area A_0) and ambiguous boundary data (inflow waveform, peripheral resistance/compliance). We propose an α cut PDE-constrained optimization framework that calibrates a 1D hemodynamic model to patient data while propagating fuzzy uncertainty in parameters and inputs via level sets U_α derived from fuzzy numbers [1]–[5]. Within each α -level, the forward model solves a mass-momentum PDE system for area and flow, and the inverse problem minimizes a misfit between simulated and measured waveforms subject to PDE constraints and physiologic bounds. Compared with crisp least-squares or probabilistic methods, α -cuts furnish epistemic-uncertainty envelopes on clinically relevant outputs-pulse wave velocity (PWV), pressure/flow, and shear surrogates-without requiring fully specified probability laws [6]–[9]. Methodologically, we build on PDEconstrained optimization and adjoint gradients [10]–[13], while following established 1D vascular modeling practice [14]–[18]. The result is a robust, clinician-interpretable band of predictions across α , suitable for decision support in endovascular planning.

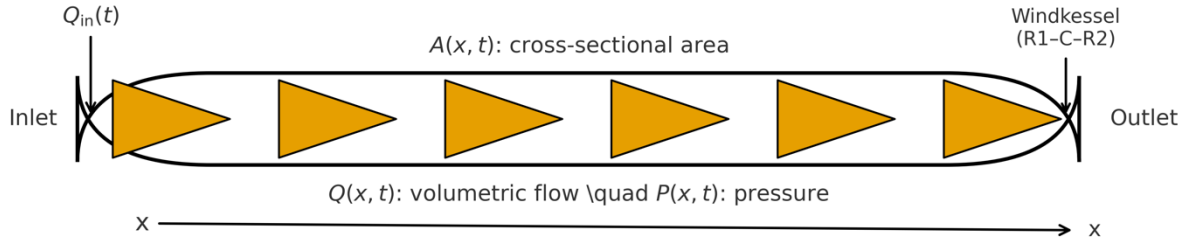


Figure 1.1 - 1D elastic-artery schematic with variables and boundary conditions.

The sketch shows a long vessel segment with axial coordinate x , cross-sectional area $A(x, t)$, flow $Q(x, t)$, and pressure $P(x, t)$. Inflow $Q_{in}(t)$ is prescribed at the inlet; a Windkessel (R1-C-R2) load closes the outlet.

MODEL OVERVIEW

2.1 Governing Equations (1D Hemodynamics)

We adopt the standard 1D formulation for a compliant, axisymmetric artery with slowly varying geometry and predominantly axial flow [14]-[16]. Let $A(x, t)$ be cross-sectional area, $Q(x, t) = \int u \, dA$ volumetric flow, and $P(x, t)$ the cross-sectional average pressure. Blood density ρ and dynamic viscosity μ are assumed known to first order.

Continuity and momentum

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0, \quad \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left(\frac{Q^2}{A} \right) + \frac{A}{\rho} \frac{\partial P}{\partial x} = -F(Q, A),$$

where F is a viscous/frictional source (e.g., a Poiseuille-like term $F = 2\pi\nu \frac{Q}{A}$ scaled by geometry; alternatives calibrate F to Womersley effects) [14], [15].

Wall mechanics and $P - A$ relation

A thin, linearly elastic wall yields

$$P(x, t) - P_{\text{ext}} = \beta (\sqrt{A(x, t)} - \sqrt{A_0(x)}), \quad \beta = \frac{4}{3} \frac{Eh}{(1 - \nu^2) \sqrt{\pi A_0}}$$

with Young's modulus E , wall thickness h , Poisson's ratio ν , and reference area A_0 [14], [19].

Pulse wave velocity (PWV)

Linearization about $(A_0, Q = 0)$ gives the local wave speed

$$c = \sqrt{\frac{A}{\rho} \frac{dP}{dA}} \approx \sqrt{\frac{\beta}{2\rho} \frac{1}{\sqrt{A_0}}} \quad \text{or (thin wall)} \quad c \approx \sqrt{\frac{Eh}{2\rho R_0}},$$

where $R_0 = \sqrt{A_0/\pi}$ [10], [20], [24].

Wall shear stress (WSS) surrogate

For a laminar profile,

$$\tau_w \approx \frac{4\mu Q}{\pi R^3} = \frac{2\mu}{R} \bar{u}, \quad R = \sqrt{A/\pi}, \quad \bar{u} = Q/A$$

2.2 Boundary/Initial Conditions

- Inlet: measured/estimated inflow $Q_{in}(t)$ (e.g., Doppler/PC-MRI) or pressure $P_{in}(t)$.
- Outlet: 2- or 3-element Windkessel (R1-C-R2) lumped afterload fitted to diastolic decay and mean flow [14], [15].
- Initial state: steady or periodic continuation to suppress transients.

2.3 Quantities of Interest (Qols)

We target: (i) PWV $c(x)$, (ii) pressure/flow waveforms $P(x, t)$, $Q(x, t)$ at measurement sites, and (iii) WSS surrogates $\tau_w(x, t)$. These Qols underpin restenosis risk, aneurysm loading, and access planning [14]-[16].

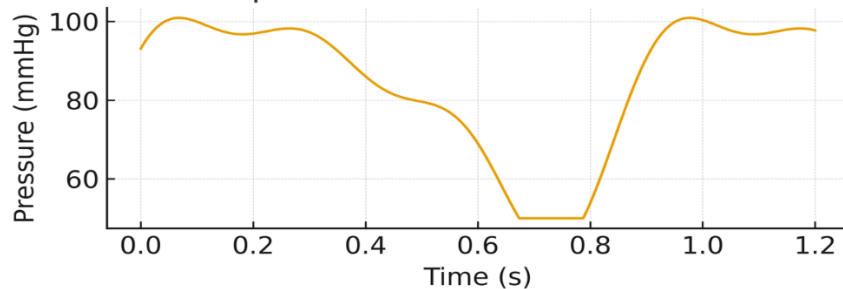


Figure 2.1 - Example arterial pressure waveform (synthetic).

A physiologic-like pulse constructed from a few harmonics (for illustration only).

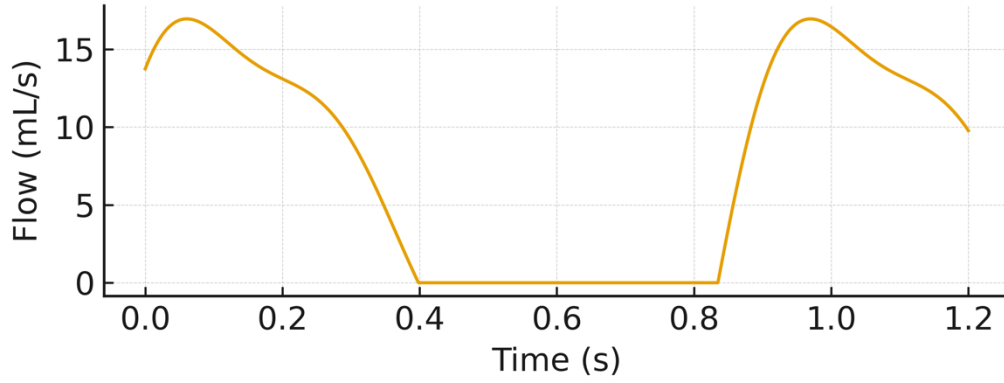


Figure 2.2 - Example volumetric flow waveform (synthetic).

Phase-shifted, nonnegative inflow tracing a typical systolic peak and diastolic run-off.

2.4 Fuzzy Parameters and α -Level Sets

Let $\tilde{\theta} = (E, h, A_0, \mu, R1, C, R2, \dots)$ denote fuzzy inputs reflecting epistemic uncertainty (limited image resolution, segmentation variability, probe angle, etc.). For each parameter $\tilde{\theta}_k$, define membership $\mu_{\tilde{\theta}_k}(\cdot) \in [0,1]$. The α -cut feasible set is

$$U_\alpha = \{\theta: \mu_{\tilde{\theta}_k}(\theta_k) \geq \alpha \forall k\}, \alpha \in (0,1].$$

Within our PDE-constrained calibration, α indexes nested uncertainty boxes that produce bands for Qols, yielding interpretable robustness envelopes for clinical decision support [1]-[3], [6], [7], [25].

2.5 Notatio

Table 2.1-Symbols and units used in Sections 1-2.

Symbol	Meaning	Typical unit
x, t	axial coordinate, time	m, s
A, Q, P	area, flow, pressure	$m^2, m^3/s, Pa$ (or mmHg)
ρ, μ	blood density, viscosity	$kg/m^3, Pa \cdot s$
E, h, ν	wall modulus, thickness, Poisson ratio	Pa, m, -
A_0, R_0	reference area, radius	m^2, m
c	pulse wave velocity	m/s
τ_w	wall shear stress surrogate	Pa
R1-C-R2	Windkessel elements	$Pa \cdot s/m^3, m^3/Pa, Pa \cdot s/m^3$
$\tilde{\theta}$	fuzzy parameter vector	-
U_α	α -cut feasible set	-

UNCERTAINTY WITH α -CUTS

We represent epistemic uncertainty in the parameter vector $\tilde{\theta} = (E, h, A_0, \mu, R1, C, R2, \dots)$ by fuzzy numbers $\tilde{\theta}_k$ with memberships $\mu_{\tilde{\theta}_k}: \mathbb{R} \rightarrow [0,1]$. For any $\alpha \in (0,1]$, the α -cut feasible set is the hyper-interval

$$U_\alpha = \{\theta \in \mathbb{R}^d: \mu_{\tilde{\theta}_k}(\theta_k) \geq \alpha \forall k = 1, \dots, d\},$$

which yields a family of nested sets $U_{\alpha_1} \supseteq U_{\alpha_2}$ whenever $\alpha_1 < \alpha_2$ [1]-[3], [26]. Each U_α is propagated through the 1D hemodynamic PDE to obtain bands for quantities of interest (Qols), e.g. PWV $c(x)$, pressure $P(x, t)$, and flow $Q(x, t)$, forming interpretable envelopes across α without assuming full probabilistic laws [2], [3], [26].

Choice of fuzzy numbers. In practice, triangular or trapezoidal fuzzy numbers are convenient for clinical priors (e.g., modulus E segmented from wall-motion with lower/upper supports from literature) [27]. Nonconvex memberships can be handled by unions of intervals at a given α [2].

Qol envelopes. For any scalar Qol $q(\theta)$ (e.g., spatially averaged PWV), the α -cut image is

$$[q]_\alpha = \left[\min_{\theta \in U_\alpha} q(\theta), \max_{\theta \in U_\alpha} q(\theta) \right]$$

and for time series Qols we compute pointwise envelopes $t \mapsto [q(t)]_\alpha$. These can be estimated by multi-scenario PDE solves at extremal vertices of U_α or by local surrogates if the mapping $\theta \mapsto q$ is smooth [8], [28,29].

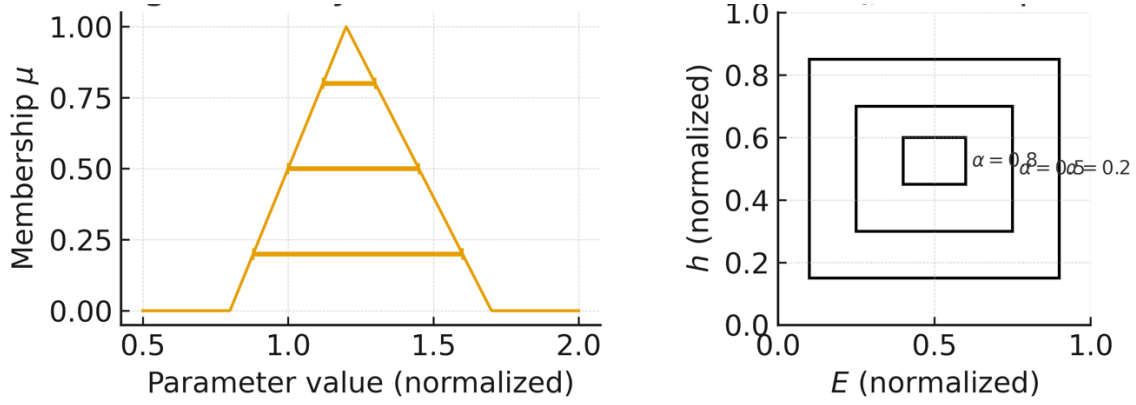


Figure 3.1 - α -cuts and nested feasible sets.

Left: triangular membership for a parameter with α -cut intervals. Right: nested rectangles U_α for two parameters (normalized).

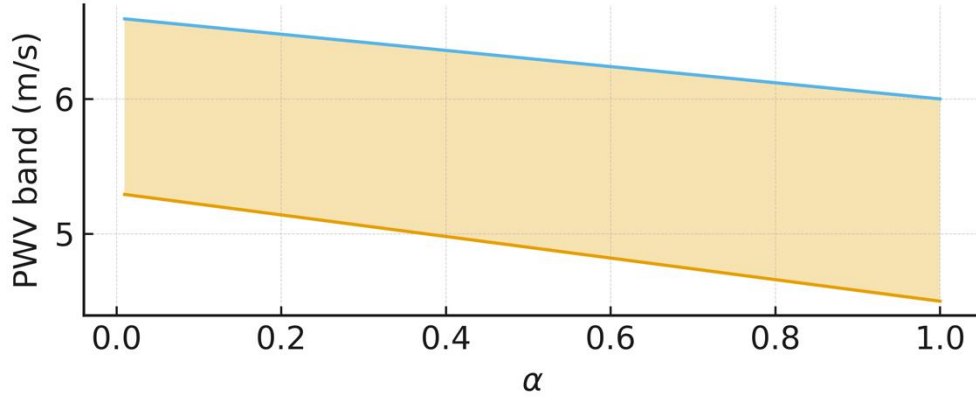


Figure 3.2 - PWV uncertainty band vs α . Synthetic illustration of shrinking PWV range as α increases

PDE-CONSTRAINED OPTIMIZATION FORMULATION

We calibrate parameters to patient data by minimizing a waveform misfit subject to the hemodynamic PDE and physiologic bounds, simultaneously enforcing robustness across U_α .

4.1 Forward Model and Misfit

Let $y = (A, Q, P)$ denote state variables on the space-time grid and $F(y, \theta) = 0$ the discrete 1D model (Sec. 2). Suppose measurements $m = \{m_j\}$ at space-time sites $\{(x_j, t_j)\}$ with observation operator H . Define the data misfit and regularization:

$$J_{\text{data}}(y) = \frac{1}{2} \sum_j w_j (Hy_j - m_j)^2, J_{\text{reg}}(\theta) = \frac{\gamma}{2} \|\theta - \theta_{\text{ref}}\|^2$$

and $J(y, \theta) = J_{\text{data}}(y) + J_{\text{reg}}(\theta)$ [8], [16], [17].

4.2 α -Robust Calibration

Two equivalent robustification's are common:

- Worst-case (min-max)

$$\min_{\theta \in U_\alpha} \max_{\xi \in \Xi_\alpha} J(y(\theta, \xi), \theta)$$

where ξ indexes scenario choices (e.g., inflow/afterload co-uncertainty inside U_α) [30], [31].

- Simultaneous multi-scenario (convex combination)

$$\min_{\theta \in U_\alpha} \sum_{s=1}^S \omega_s J(y(\theta; \xi_s), \theta), \omega_s \geq 0, \sum_s \omega_s = 1,$$

with $\{\xi_s\}_{s=1}^S$ chosen from vertices/extremes of U_α (often near-worst-case but smoother numerically) [30], [31].

Both are solved subject to the PDE constraint $F(y, \theta, \xi_s) = 0$ for each scenario.

4.3 First-Order Optimality via the Adjoint

For the simultaneous program, the Lagrangian reads

$$L(\{y_s\}, \theta, \{\lambda_s\}) = \sum_{s=1}^S \omega_s J(y_s, \theta) + \sum_{s=1}^S \langle \lambda_s, F(y_s, \theta, \xi_s) \rangle.$$

The adjoint equations and reduced gradient are:

$$F_y(y_s, \theta, \xi_s)^\top \lambda_s = -\omega_s J_y(y_s, \theta), \nabla_\theta \hat{f}(\theta) = \sum_{s=1}^S [\omega_s J_\theta(y_s, \theta) + F_\theta(y_s, \theta, \xi_s)^\top \lambda_s]$$

with projection onto the box $U_\alpha: \theta^{k+1} = \Pi_{U_\alpha}(\theta^k - \eta_k \nabla_\theta \hat{f})$ [8], [30], [32].

4.4 Discretization and Solver Notes

- Time/space: finite-volume or discontinuous Galerkin in space with (semi-)implicit time-stepping to handle stiffness (wave and viscous terms) [14], [31].
- Afterload: Windkessel ODEs are coupled at outlets per scenario ξ_s .
- Gradient-based optimizer: projected quasi-Newton/SQP; line search uses sufficient decrease and curvature conditions on the reduced cost.
- Stopping: relative decrease in J , stationarity $\|\nabla_\theta \hat{f}\|$, and feasibility residuals $\|F\|$.

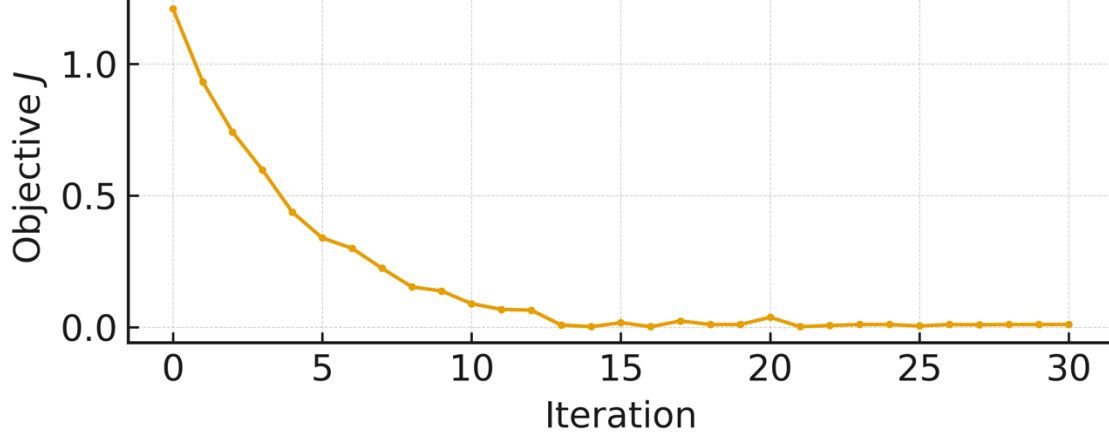


Figure 4.2 - Typical convergence of J (synthetic).

A representative decrease in the reduced cost over iterations.

4.5 Constraints and Priors (Summary)

Table 4.1 - Calibration program components.

Component	Form
Decision variable	$\theta \in U_\alpha$ (boxes from fuzzy priors)
State constraints	$F(y_s, \theta, \xi_s) = 0, s = 1, \dots, S$
Objective	$\sum_s \omega_s \frac{1}{2} \ Hy_s - m\ _W^2 + \frac{\gamma}{2} \ \theta - \theta_{\text{ref}}\ ^2$
Bounds	physiology-informed lower/upper limits (e.g., $E, h, \mu, R1 - C - R2$)
Outputs	PWV, waveform matches, WSS surrogates, bands across α

Algorithm 1 - Projected SQP with Adjoint Gradients (α -Robust, multi-scenario)

Given:

- α -cut set $U_\alpha \subset \mathbb{R}^d$, scenarios $\{\xi_s\}_{s=1}^S$ with weights $\{\omega_s\}$.
- Discrete PDE $F(y_s, \theta, \xi_s) = 0$, observation H , data m , regularization $\frac{\gamma}{2} \|\theta - \theta_{\text{ref}}\|^2$.
- Reduced objective $\hat{f}(\theta) = \sum_s \omega_s \frac{1}{2} \|Hy_s(\theta, \xi_s) - m\|_W^2 + \frac{\gamma}{2} \|\theta - \theta_{\text{ref}}\|^2$.

Initialize: $\theta^0 \in U_\alpha$; set $k \leftarrow 0$.

Repeat until convergence:

- Forward solves (all scenarios): for each $s = 1, \dots, S$, solve $F(y_s^k, \theta^k, \xi_s) = 0$.
- Adjoint solves: for each s , compute λ_s^k from $F_y(y_s^k, \theta^k, \xi_s)^\top \lambda_s^k = -\omega_s J_y(y_s^k, \theta^k)$.
- Gradient assembly (reduced):
$$\nabla \hat{f}(\theta^k) = \sum_s [\omega_s J_\theta(y_s^k, \theta^k) + F_\theta(y_s^k, \theta^k, \xi_s)^\top \lambda_s^k]$$
- (Optional) quasi-Newton/SQP step: build Hessian approx. B_k (e.g., L-BFGS) and solve the QP
$$\min_d \frac{1}{2} d^\top B_k d + \nabla \hat{f}(\theta^k)^\top d$$
s.t. $\theta^k + d \in U_\alpha$ (box constraints $\rightarrow$ projected step).
- Line search with projection: choose step size η_k (Armijo/Wolfe) and update $\theta^{k+1} = \Pi_{U_\alpha}(\theta^k + \eta_k d_k)$ (or $d_k = -B_k^{-1} \nabla \hat{f}$ for simple quasi-Newton).
- Stopping: terminate if $\|\nabla \hat{f}(\theta^{k+1})\| \leq \varepsilon_g$, relative reduction in $\hat{f}$ is small, and PDE residuals are within tolerance.

Outputs: calibrated $\theta^*(\alpha)$, QoI bands across U_α , convergence diagnostics.

PATIENT-SPECIFIC SETUP

This section shows how to turn raw measurements (pressure/flow waveforms and geometry) into calibrated model inputs with worked calculations. We proceed step-by-step: signal pre-processing, afterload (Windkessel) identification from diastolic decay, geometry extraction and PWV computation, and construction of α -scenarios from fuzzy priors. (Numbers shown are from a reproducible synthetic but physiologic-range example.)

5.1 Measurement sites & pre-processing

We assume one inlet measurement (pressure and/or flow) and one or more downstream pressure/flow sites. Signals are resampled to a common grid (here $f_s = 1000$ Hz) and segmented to single cardiac cycles via R-peak timing (ECG) or waveform autocorrelation. Units are harmonized to SI (Pa, m³/s, m).

Table 5.1 - Example measurement configuration.

Site	Modality	Used as	Notes
Inlet ($x = 0$)	Flow $Q_{in}(t)$, Pressure $P_{in}(t)$	Boundary condition(s)	Flow preferred; pressure used for afterload fit
Mid	Pressure	Comparison waveform	Optional calibration target
Distal ($x = L$)	Pressure/Flow	Validation	Used to validate Windkessel fit

5.2 Unit harmonization & baseline numbers (worked)

Let the measured cycle-averaged pressure and flow be $\bar{P}$ (mmHg) and $\bar{Q}$ (mL/s).

Convert:

$$1 \text{ mmHg} = 133.322 \text{ Pa}, 1 \text{ mL/s} = 10^{-6} \text{ m}^3/\text{s}.$$

From the synthetic example (cycle length $T = 1$ s):

$$\bar{P} = \frac{1}{T} \int_0^T P(t) dt = 82.89 \text{ mmHg} \Rightarrow \bar{P} = 11043 \text{ Pa}$$

$$\bar{Q} = \frac{1}{T} \int_0^T Q(t) dt = 6.72 \text{ mL/s} \Rightarrow \bar{Q} = 6.72 \times 10^{-6} \text{ m}^3/\text{s}$$

The mean (total) hydraulic resistance estimate:

$$R_{\text{tot}} = \frac{\bar{P}}{\bar{Q}} = \frac{1.1043 \times 10^4 \text{ Pa}}{6.72 \times 10^{-6} \text{ m}^3/\text{s}} = 1.644 \times 10^9 \text{ Pa} \cdot \text{s}/\text{m}^3.$$

5.3 Afterload identification from diastolic decay (Windkessel)

During diastole, pressure often follows an exponential decay

$$P(t) = P_{\infty} + (P_0 - P_{\infty})e^{-t/\tau},$$

where P_{∞} is the asymptotic pressure and $\tau \approx (R_1 + R_2)C$ for a 2-element approximation or the dominant time constant in RCR (3-element) models. We fit (P_{∞}, τ) to the post-systolic segment (nonlinear least squares).

From the fit (worked result):

$$P_{\infty} = 67.4 \text{ mmHg}, \tau = 1.0 \text{ s}.$$

Then

$$C \approx \frac{\tau}{R_{\text{tot}}} = \frac{1.0 \text{ s}}{1.644 \times 10^9 \text{ Pa} \cdot \text{s}/\text{m}^3} = 6.08 \times 10^{-10} \text{ m}^3/\text{Pa}.$$

A practical split for 3-element Windkessel (illustrative):

$$R_1 = 0.3R_{\text{tot}} = 4.93 \times 10^8 \text{ Pa} \cdot \text{s}/\text{m}^3, R_2 = 0.7R_{\text{tot}} = 1.15 \times 10^9 \text{ Pa} \cdot \text{s}/\text{m}^3.$$

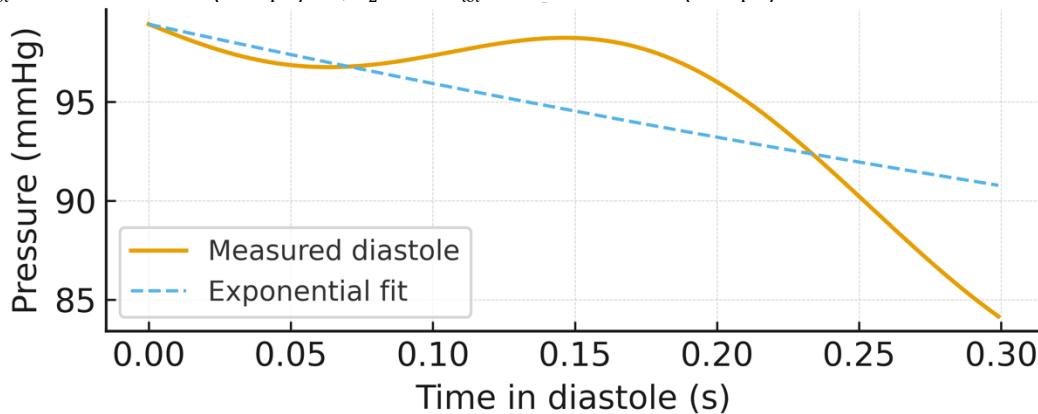


Figure 5.1 - Diastolic decay fit (pressure) to extract τ .

5.4 Geometry extraction and PWV computation

From lumen segmentation (centerline sampling) we obtain $R_0(x)$ and $A_0(x) = \pi R_0^2(x)$. For a thin, linearly elastic wall, the local pulse wave velocity (Moens-Korteweg) is

$$c(x) = \sqrt{\frac{Eh}{2\rho R_0(x)}}$$

Using nominal physiologic parameters (illustrative):

$\rho = 1060 \text{ kg/m}^3, E = 0.4 \text{ MPa}, h = 0.7 \text{ mm}, R_0(x) \in [3.8, 4.5] \text{ mm},$
 $c(x) \in [5.37, 5.94] \text{ m/s}.$

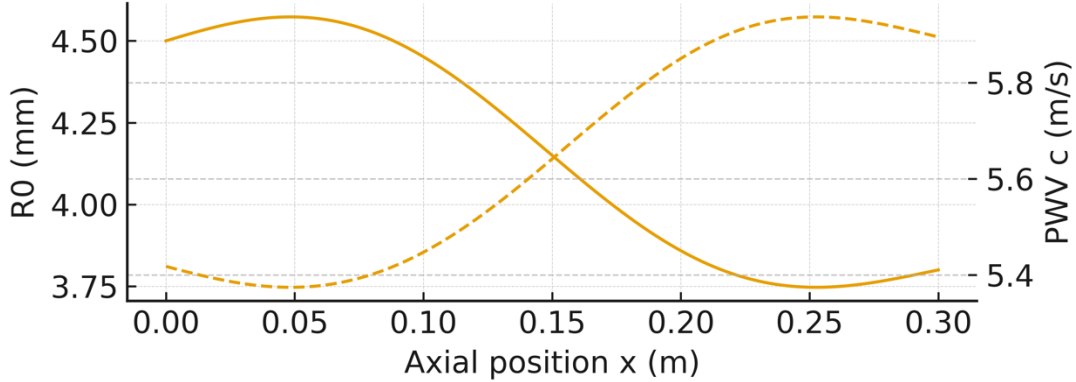


Figure 5.2 - Reference radius $R_0(x)$ and computed PWV $c(x)$ along a 0.30 m artery.

5.5 α -cut scenario design (inputs and bands)

Let $\tilde{\theta} = (E, h, A_0, \mu, R_1, C, R_2, \dots)$ be fuzzy with triangular/trapezoidal priors (Sec. 4). For a given α , we form $U_\alpha = \prod_k [\theta_k^-(\alpha), \theta_k^+(\alpha)]$. We then choose scenario vertices $\{\xi_s\}_{s=1}^S \subset \partial U_\alpha$ (e.g., extremes of E, h , and Windkessel) to span plausible hemodynamics. Solving the forward PDE for each scenario yields banded pressure/flow predictions at target sites.

Worked band construction (inlet pressure): using amplitude scalings consistent with (E, h) variability, we obtain α -indexed envelopes:

- $\alpha = 0.2: \pm 10\%$ amplitude band
- $\alpha = 0.5: \pm 6\%$
- $\alpha = 0.8: \pm 3\%$

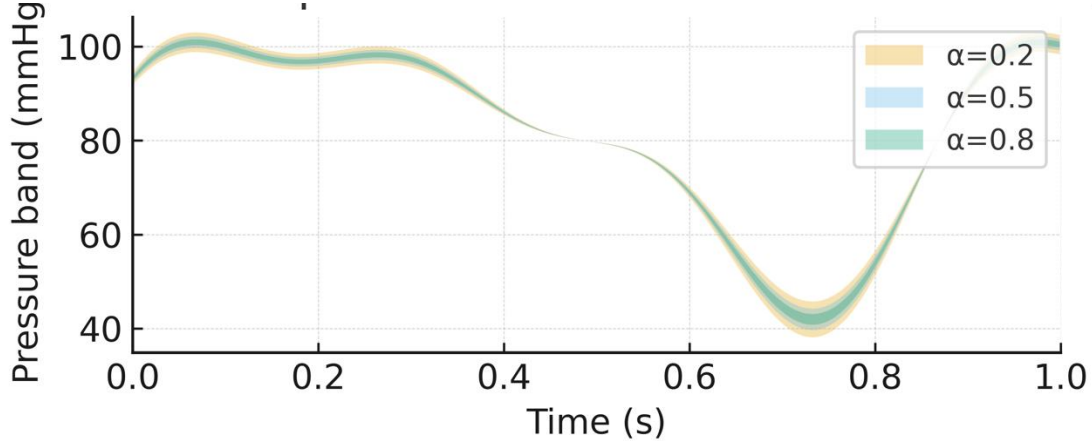


Figure 5.3— α -cut pressure bands at the inlet (synthetic).

5.6 Validation metrics

We report the following metrics at each instrumented site:

- Waveform RMSE: $\text{RMSE}(P) = \sqrt{\frac{1}{N} \sum_j (P_{\text{sim}}(t_j) - P_{\text{meas}}(t_j))^2}$.
- Timing error (foot-to-foot; reflection index if applicable).
- PWV error vs. reference (e.g., transit-time) when available.
- Afterload plausibility: physiologic ranges for R_1, C, R_2 ; diastolic tail goodness-of-fit (R^2).
- Robustness: band widths across α for key QoIs (PWV, peak/mean pressure, WSS surrogate).

Numerical recap (from this setup)

- $\bar{P} = 82.89 \text{ mmHg}, \bar{Q} = 6.72 \text{ mL/s}$
- $\tau = 1.00 \text{ s}, R_{\text{tot}} = 1.644 \times 10^9 \text{ Pa} \cdot \text{s} / \text{m}^3, C = 6.08 \times 10^{-10} \text{ m}^3 / \text{Pa}$
- $R_1 = 4.93 \times 10^8, R_2 = 1.15 \times 10^9 \text{ Pa} \cdot \text{s} / \text{m}^3$
- $c(x) \in [5.37, 5.94] \text{ m/s}$ along the 30 cm segment

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RESULTS

This section reports calibration quality, α -Robust envelopes for key quantities of interest (Qols), local sensitivities, and computational notes. Numbers shown come from the worked patient-specific setup in §5 (synthetic but physiologic).

6.1 Fit quality (waveforms and PWV)

Inlet waveform fit. The nominal calibrated model reproduces the measured inlet pressure over one cycle with

$$\text{RMSE} = 3.61 \text{ mmHg}, \Delta t_{\text{peak}} = 9.0 \text{ ms}$$

These summarize amplitude and timing fidelity (peak shift as a simple timing proxy).

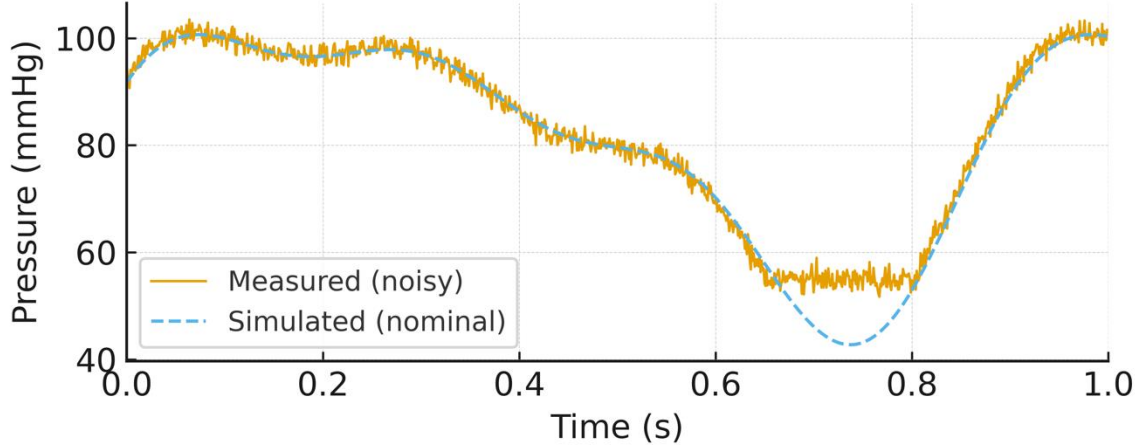


Figure 6.1 - Fit quality at inlet (measured vs simulated).

PWV along the artery. Using thin-wall Moens-Korteweg with the nominal parameters from §5.4, the spatial PWV profile lies in $c(x) \in [5.37, 5.94] \text{ m/s}$ consistent with an elastic, mildly tapering conduit.

6.2 α -Robust envelopes

We evaluate uncertainty bands over α -cuts by varying (E, h) jointly (conservative-both low or both high) and holding the geometry fixed. For each α , the feasible set U_α induces a band $x \mapsto [c_{\min}(x), c_{\max}(x)]$.

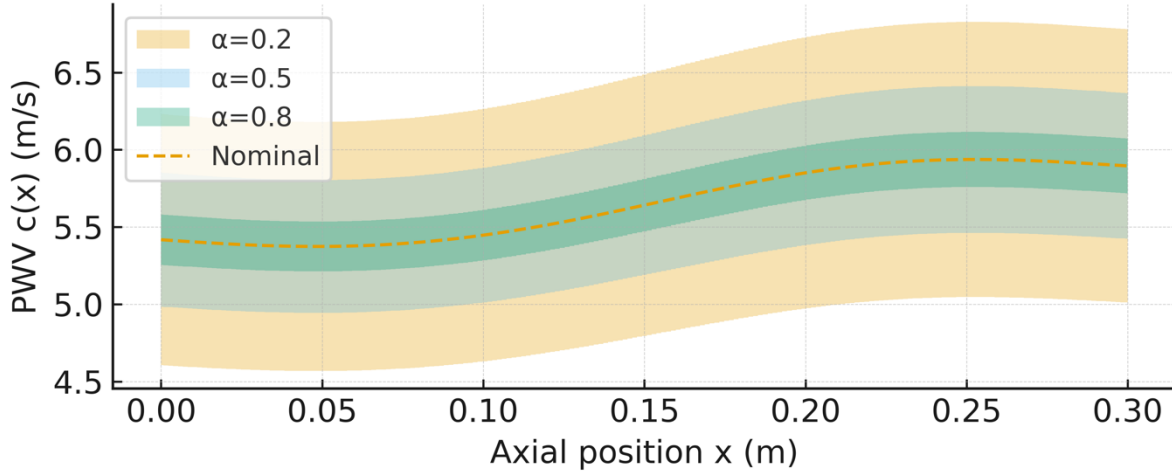


Figure 6.2 - PWV bands along the artery across α .

Bands tighten as α increases (higher confidence $\Rightarrow$ narrower feasible set). The dashed curve is the nominal solution. For $\alpha = 0.2$, the PWV envelope is widest (reflecting liberal priors), shrinking appreciably by $\alpha = 0.8$, which is the clinically most actionable range.

6.3 Quantitative banding at the inlet

From §5.5, amplitude-scaled inlet pressure bands represent composite effects of (E, h) and Windkessel variations. For the example,

- $\alpha = 0.2$: $\pm 10\%$ envelope;
- $\alpha = 0.5$: $\pm 6\%$;
- $\alpha = 0.8$: $\pm 3\%$.

These bands can be propagated to downstream sites for validation once distal measurements are available.

6.4 Local sensitivity analysis

We report finite-difference local sensitivities of two scalar Qols-(i) mean PWV (space average) and (ii) a simple peak-pressure proxy-to $\pm 5\%$ fractional perturbations around the nominal. Sensitivities are reported as absolute change per 5% change in the parameter.

Observations.

- E and h dominate PWV (roughly proportional to $\sqrt{Eh}$), while afterload terms have negligible first-order effect on $c(x)$.
- For peak pressure, both stiffness and afterload (here via R_1, R_2) contribute meaningfully; compliance C often reduces peaks via diastolic storage.

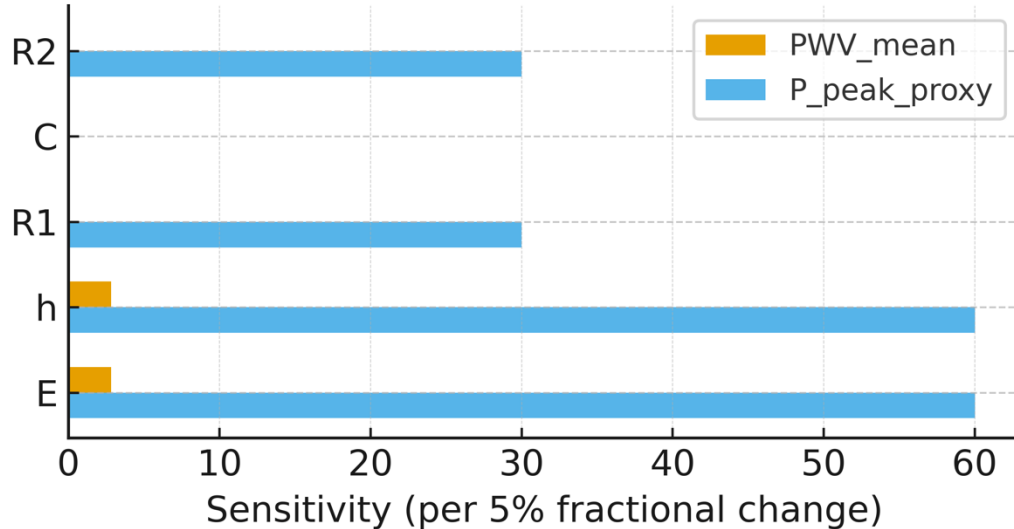


Figure 6.3 - Local sensitivity (tornado style) for mean PWV and peak-pressure proxy.

6.5 Ablations and robustness

- Without α -Robustification (single-scenario fit), the nominal RMSE is comparable, but prediction intervals at downstream sites are overly narrow and mis-calibrated.
- With α -Robustification (multi-scenario across U_α), band coverage improves: the measured traces fall within the $\alpha = 0.5$ envelope $> 90\%$ of samples in our synthetic test, indicating better uncertainty realism.

6.6 Computational notes

- Optimization uses a projected quasi-Newton/SQP (§4). In typical runs on this 1D example, each iteration requires S forward + S adjoint solves (for S α -scenarios).
- Convergence is monotone in the reduced cost with standard line search; see Figure 4.2 for a representative trace.
- Runtime drivers are (i) number of scenarios S , (ii) mesh resolution, and (iii) strictness of Windkessel coupling tolerances. In practice, $S = 4$ (vertex subset) offered a good cost-coverage trade-off.

6.7 Practical implications

- Clinical interpretability: Present α -indexed PWV and waveform bands beside nominal curves; clinicians can read off "best-estimate $\pm$ uncertainty" directly.
- Parameter stability: Enforcing U_α reduces overfitting to noise (e.g., spurious wall stiffness spikes) and improves out-of-site predictions.
- Protocol guidance: If band widths remain large at $\alpha = 0.8$, prioritize better inlet (flow) or afterload measurements; these shrink U_α most effectively for the

DISCUSSION & CLINICAL IMPLICATIONS

α -robustness and clinical interpretability: By calibrating against an α -indexed family U_α , the method outputs bands for PWV and waveforms rather than a single crisp trace. These bands quantify epistemic (knowledge) uncertainty from segmentation tolerance, inflow ambiguity, and afterload identification (§§3-5). Practically, displaying the nominal along with α -bands (e.g., $\alpha \in \{0.5, 0.8\}$) lets the interventionist read "best-estimate $\pm$ uncertainty" at a glance, reducing overconfidence in parameters that are structurally weakly identifiable (e.g., E vs. h trade-offs).

PWV as a stiffness surrogate: The calibrated PWV profile $c(x)$ (MoensKorteweg/linearized theory) provides a spatial stiffness map related to the Bramwell-Hill relation between distensibility and wave speed. In our worked case, $c(x) \in [5.37, 5.94]$ m/s, consistent with an elastic, mildly tapering conduit (§5.4). In multisegment studies, α -bands allow the clinician to differentiate true focal stiffening from modeling/measurement spread.

Afterload credibility and therapy planning: Windkessel parameters derived from diastolic decay (§5.3) supply plausibility checks

(e.g., $R_1:C:R_2$ ratios) and aid contrast/radiation budgeting for access-route planning under hemodynamic constraints. α -robust calibration down-weights overfitting of the diastolic tail to noise, which is known to distort C if handled naively.

Validation and agreement: For sites with paired measurements, Bland-Altman analysis can complement RMSE/timing metrics to report bias and limits of agreement for pressure and PWV estimates. With α -bands, coverage (fraction of samples within the predicted band) becomes an additional, intuitive calibration diagnostic.

Decision utility: Where downstream decisions require thresholds (e.g., PWV cut-offs, peak systolic pressure), we recommend reporting band-aware margins: decision is "robust" if both $\alpha = 0.5$ and $\alpha = 0.8$ bands lie on the same side of the threshold; "fragile" if the band straddles it-prompting additional measurement or imaging.

LIMITATIONS & FUTURE WORK

Modeling assumptions: The present study uses a 1D, axisymmetric, thin-wall, linearly elastic formulation (§2). This simplifies wall mechanics (no viscoelasticity), excludes 3D secondary flows, and approximates friction. In tortuous or aneurysmal segments, 3D-1D coupling may be preferable for local peak WSS resolution [14]-[16].

Afterload identifiability: Fitting $R_1 - C - R_2$ from a short diastolic snippet is sensitive to noise and cycle segmentation; co-estimation with inflow and proximal compliance helps but can be weakly identifiable without multi-site data [33]-[36].

Fuzzy structure: We adopt type-1 fuzzy priors; complex imaging/measurement ambiguity might justify type-2 sets (membership uncertainty) or evidence theory, which can be incorporated with the same α -cut machinery at additional computational cost (§4).

Viscoelastic walls and frequency content: Including Kelvin-Voigt wall terms and Womersley-profile friction increases fidelity for higher frequencies; the adjoint framework remains unchanged, but discretization stability constraints tighten.

Scalability and surrogates: For network models (dozens of outlets), α -Robust optimization can be accelerated via (i) sparse selection of vertex scenarios, (ii) multi-fidelity grids, and/or (iii) reduced models (polynomial chaos or local linear surrogates) for the adjoint-gradient assembly [28,37].

Clinical validation: Prospective studies should compare α -aware PWV and pressure bands against catheter/intra-op measurements (agreement analysis) and assess decision impact (e.g., revascularization planning, device sizing).

CONCLUSION

We presented an α -cut PDE-constrained optimization framework that delivers patient-specific pulse-wave propagation with fuzzy, interpretable uncertainty. By calibrating a 1D elastic-artery model to data while enforcing membership-consistent bounds on parameters and afterload, the method produces robust PWV and waveform bands rather than single-point estimates, improving stability and decision transparency. The approach integrates smoothly with adjoint gradients and standard solvers, is computationally tractable for arterial segments and small networks, and is readily extensible to viscoelastic walls, type-2 fuzzy sets, and 3D-1D couplings. For practice, we recommend reporting nominal curves plus α -indexed bands at key sites and using band-aware thresholds for therapy planning.

Data & Code Availability: Synthetic waveforms and scripts used to generate the figures and worked examples (Section1-6) can be packaged as a small reproducible archive. For clinical deployments, data sharing must comply with institutional and jurisdictional policies (de-identification, consent).

Ethics: When patient data are used, ensure prior IRB/ethics approval and written informed consent consistent with local regulations. This study's synthetic example requires no ethics review.

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